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2d Topological Quantum Field Theories and Frobenius Algebras

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Abstract: As background-independent toy models of quantum gravity, low-energy effective field theories of states in condensed matter physics, and a recipe for topological invariants of closed manifolds, Topological Quantum Field Theories (TQFTs) have become integral to studies in algebraic topology and mathematical physics over the last several decades. In this brief note, we present the correspondence between 2d TQFTs and Frobenius algebras. We will first introduce TQFTs from mathematical and physical perspectives. We then discuss aspects of the category of 2-cobordisms, followed by the Atiyah-Segal axioms for TQFTs. We proceed by discussing commutative Frobenius algebras and presenting a proof (sketch) of the equivalence between 2d TQFTs and commutative Frobenius algebras. We conclude by pointing out some general aspects of the theory.

Contents

[1]Special thanks to Ajay who oversaw great portions and details of the paper.

1 Introduction

In the 1980s, due to efforts lead by E. Witten and M. Atiyah among others, intricate links between quantum physics and topology were beginning to be realized. Perhaps one of the most monumental works of this type was Witten's [1] where he derived various aspects of Morse theory from considerations of supersymmetric quantum mechanics, and established a correspondence between supersymmetric quantum field theories (QFTs) and the differential geometry of infinite dimensional manifolds. Such studies linking quantum physics, topology, and geometry led to a new perspective in studies of low-dimensional manifolds[1]. In [2], Atiyah presented a set of axioms for a TQFT, a QFT that is intrinsically topological (based on physical models discussed by Witten in [3]). It soon became clear that TQFTs constitute topological invariants of closed manifolds, and were of much relevance to aspects of topology like knot invariants and the classification of manifolds.

An n-dimensional TQFT is a rule $\mathcal{A}$ that associates to each closed oriented smooth $(n-1)$-manifold Σ a vector space $\Sigma\mathcal{A}$[2], and to each oriented smooth n-manifold M such that $\partial M = \Sigma$ a vector $M\mathcal{A} \in \partial M\mathcal{A}$. Our object of interest is $\mathbf{2TQFT}_{\Bbbk}$, the category with objects 2-dimensional TQFTs. The main result presented in this note is the classification of the structure of 2TQFTs first noted by R. Dijkgraaf in [4].

Theorem 1.1. *For any ground field $\Bbbk$, there is an equivalence*

$$\mathbf{2TQFT}_{\Bbbk} \simeq \mathbf{cFA}_{\Bbbk},$$

by sending each TQFT to its valuation on the circle. Here $\mathbf{cFA}_{\Bbbk}$ is the category of commutative Frobenius $\Bbbk$-algebras.

The rest of this note is focused on developing the required framework to study TQFTs and establish this correspondence. Much of the content here has been adapted from [5].

2 Cobordisms

2.1 The Category 2Cob

We always consider smooth compact manifolds (and hence omit 'smooth' and 'compact' from hereon). By a closed manifold we mean a manifold without boundary. Manifolds

[1]One associates to such a manifold a suitable infinite-dimensional manifold, and studies the associated QFT. However, establishing general QFT on an axiomatic footing is a long-standing open problem.

[2]We adopt the notation where the image of a functor is denoted by a postfix (with $\mathcal{A}$) rather than a prefix

with boundary are denoted by roman letters like M, while those without are denoted by greek letters like Σ.

Definition 2.1 (In boundaries and Out-boundaries)**.** *Let M be an oriented n-manifold, let Σ be a closed submanifold of M that is a connected component of ∂M. For $x \in \Sigma$, we say $w \in T_xM$ is positive normal if $\{w\}$ adjoined at the last position of a positively oriented basis of $T_x\Sigma$ is a positively oriented basis for T_xM. If a positive normal of Σ points into (out from) M, we say Σ is an in-boundary (out-boundary) of M.*

Definition 2.2 (Cobordisms)**.** *For two closed oriented $(n-1)$-manifolds Σ_0 and Σ_1, a cobordism (or n-cobordism) from Σ_0 to Σ_1 is an oriented manifold M with maps $\Sigma_0 \to M \leftarrow \Sigma_1$ such that Σ_0 (Σ_1) maps diffeomorphically preserving (reversing) orientations to the in-boundary (out-boundary) of M. We write $M : \Sigma_0 \to \Sigma_1$ to mean that M is a cobordism.*

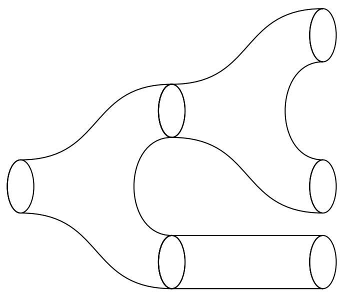
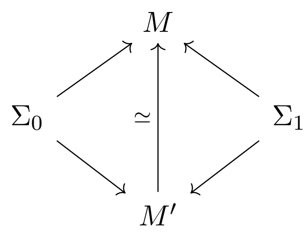


Figure 2.1: An example of a 2-cobordism

For $M, M' : \Sigma_0 \to \Sigma_1$, we say $M \sim M'$ (or that M and M' are equivalent) if there exists a diffeomorphism $M \to M'$ such that the following commutes.



The above defines an equivalence relations on cobordisms $\Sigma_0 \to \Sigma_1$. We now define the composition of two cobordisms by gluing. The main result is the following.

Theorem 2.1. *Let $M : \Sigma_0 \to \Sigma_1$ and $M' : \Sigma_1 \to \Sigma_2$ be two n-cobordisms. There exists a smooth structure on the topological manifold $MM' := M \amalg_{\Sigma_1} M'$ (termed the composition of M and M'), the gluing of M and M' along Σ_1, such that $M \hookrightarrow MM'$ and $M' \hookrightarrow MM'$ are diffeomorphisms onto their images. This structure is unique up to diffeomorphism fixing Σ_0, Σ_1, and Σ_2.*

While we refer the reader to [6] for the detailed proof, we sketch a construction for $n = 2$ here (the case of relevance). The first fact to note that the composition of two cylinders is still a cylinder and thus has a natural smooth structure given by the product (and this structure satisfies the required axioms). Now given two cobordisms $M : \Sigma_0 \to \Sigma_1$ and $M' : \Sigma_1 \to \Sigma_2$, we take Morse functions $f : M \to [0,1]$ and $f' : M' \to [1,2]$ (i.e. smooth functions with no degenerate critical points). Pick $\varepsilon > 0$ such that $[1-\varepsilon, 1]$ is regular for f (i.e. f has no critical points in this interval) and $[1, 1+\varepsilon]$ is regular for f'. Then by the regular interval theorem (see [7] for details), we see that the preimages of these intervals are each diffeomorphic to cylinders. Within $[1-\varepsilon, 1+\varepsilon]$ we are in the instance of gluing cylinders, and we can take the smooth structure of the composition to be the one induced by the smooth structure associated to this cylinder. To see that this structure is unique up to diffeomorphism fixing the respective boundaries, it is now enough to note that between smooth 2-manifolds diffeomorphisms are isotopic to homeomorphisms (this follows from the classification of 2-manifolds, see [7] for details).

Theorem 2.2. *There is a category* **nCob** *whose objects are closed oriented* $(n-1)$*-manifolds and arrows (*$\Sigma_0 \to \Sigma_1$ *for* $\Sigma_0, \Sigma_1 \in$ **nCob** *say) are equivalence classes of cobordisms from* Σ_0 *to* Σ_1*, where the composition of two classes is given by the class of the composite of representative cobordisms as defined in 2.1.*

Proof. 2.1 proves well-definition of composition. Associativity of composition follows from the associativity of pushouts in **Top** (by that we mean gluing is associative on **Top**). The identity arrow for each $\Sigma \in$ **nCob** is given by the class of the cylinder on Σ. To show that this is indeed the identity for composition, it is enough to note that every cobordism is diffeomorphic to a cylinder in a part sufficiently close to the boundary (this is again a result of the regular interval theorem), and the composite of two cylinders is again a cylinder. Details are left to the reader. □

We conclude with this important result.

Theorem 2.3. *The category* **2Cob** *is generated by the following cobordisms under gluing and disjoint union (i.e. parallel connection) and composition (i.e. the arrows in* **2Cob** *are generated upto isomorphism by these generators)*[3]*.*

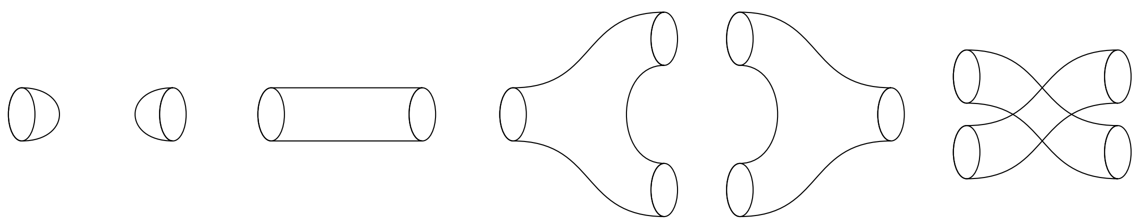


Figure 2.2: Generators for **2Cob** - labeled $(\mathbf{1},\mathbf{0})$, $(\mathbf{0},\mathbf{1})$, $(\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{2})$, and T (left to right)

[3] $(\mathbf{1},\mathbf{0})$, $(\mathbf{0},\mathbf{1})$, $(\mathbf{1},\mathbf{1})$, and $(\mathbf{1},\mathbf{2})$ are defined uniquely by the number of in- and out-boundaries (as well as the fact that they have genus 0) by the classification theorem. T is defined as the unique cobordism generated by the twist diffeomorphism $\mathbf{1} \amalg 1 \to \mathbf{1} \amalg \mathbf{1}$

Proof. We first define a skeleton for **2Cob**. Recall that a skeleton of a category is a full subcategory where no two distinct objects are isomorphic, and which contains an object from each isomorphism class. The two key facts are as follows: every closed 1-manifold is diffeomorphic to a finite disjoint union of circles (see [8] pp.208 for a proof of the classification of 1-manifolds), and two closed 1-manifolds are isomorphic in **2Cob** iff they are diffeomorphic. Now let **0** be the empty 1-manifold, **1** denote a circle Σ, and **n** denote the the n-fold disjoint union of Σ. By the above remarks, it follows that $\{\mathbf{0}, \mathbf{1}, \cdots, \mathbf{n}, \cdots\}$ is a skeleton for **2Cob**. From now on we mean this skeleton when we write **2Cob**. Now recall by the classification of surfaces that two closed connected oriented surfaces with oriented boundary are equivalent (in the sense defined above) iff they have the same genus, number of in-boundaries, and number of out-boundaries. The idea now is that given a connected cobordism $\mathbf{n} \to \mathbf{m}$ of genus g, we can build using the above generators, another connected cobordism with n in-boundaries, m out-boundaries, and genus g. By the classification of surfaces then, these cobordisms must lie in the same class. First we construct a connected cobordism $\mathbf{n} \to \mathbf{1}$ of genus 0, the in-part (using $((\mathbf{0}, \mathbf{1})), (\mathbf{1}, \mathbf{1})$, and $(\mathbf{2}, \mathbf{1})$). Then we construct a connected cobordism $\mathbf{1} \to \mathbf{1}$ of genus g (using $(\mathbf{1}, \mathbf{2})$ and $(\mathbf{2}, \mathbf{1})$). Finally we construct a connected cobordism $\mathbf{1} \to \mathbf{m}$ of genus 0 (using $(\mathbf{1}, \mathbf{0})$, $(\mathbf{1}, \mathbf{1})$, and $\mathbf{1}, \mathbf{2}$) and sequentially glue the three parts together along the appropriate boundaries. Here is an important note, by the additivity of Euler characteristic χ, the Euler characteristic of a gluing of two 2-cobordisms is simply the sum of their Euler characteristics[4]. To see how this applies, note that the in-part has $\chi = 2-2\cdot 0-n-1$, the mid-part has $\chi = 2-2g-2$, the out-part has $\chi = 2-2\cdot 0-m-1$. We then get that the total cobordism has $\chi = 2-2g-(m+n)$ as needed[5]. The details behind this construction are left to the reader, but we provide a visual example in A.1. This resolves the issue for connected cobordisms. Suppose M is a

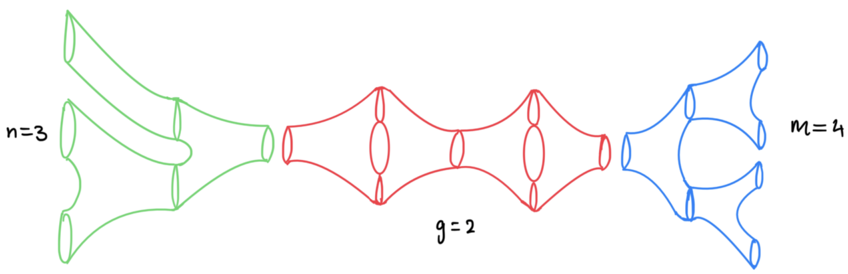


Figure 2.3: An example of the above construction with $n = 3$, $m = 4$, $g = 2$.

disconnected cobordism with two connected components M_0 and M_1. We know that M_0 and M_1 can be written as discussed above. Although we might claim that the disjoint union of these two normal forms is equivalent to M as a cobordism, this is not always true

[4] this is seen by covering the gluing $M \amalg M'$ say, by M glued with a little cylinder (recall the RVT) on the in-boundary of M' and vice versa, and applying additivity. The intersection of these sets is homotopy equivalent to a disjoint union of circles and hence has $\chi = 0$

[5] We've used $\chi = 2 - 2g - b$ for an orientable manifold of genus g with b boundary components

because of the ordering of the boundaries[6] of M_0 and M_1. But to resolve this, we may compose with the twist cobordism finitely many times to permute the boundaries and then take disjoint unions of cobordisms. Since M is compact, it will have a finite number of connected components, and we may iteratively apply this process to write M in terms of the generators in 2.2. □

2.2 Symmetric Monoidal Functors

Definition 2.3 (Symmetric Monoidal Functors from **2Cob** to $\mathbf{Vect}_{\Bbbk}$)**.** *For* $\mathbf{m}, \mathbf{n} \in \mathbf{2Cob}$ *let* $\tau_{\mathbf{m},\mathbf{n}} : \mathbf{m} \amalg \mathbf{n} \to \mathbf{n} \amalg \mathbf{m}$ *be the unique class given by the interchange of factors, whose existence is guaranteed by the fact that* $\amalg$ *is co-product on* **2Cob** *(existence and uniqueness are left as exercises to the reader). Let* $\mathcal{A} : \mathbf{2Cob} \to \mathbf{Vect}_{\Bbbk}$ *be a functor. Let* $\tau_{\mathbf{m}\mathcal{A},\mathbf{n}\mathcal{A}} : \mathbf{m}\mathcal{A} \otimes \mathbf{n}\mathcal{A} \to \mathbf{n}\mathcal{A} \otimes \mathbf{m}\mathcal{A}$ *be the unique linear map given by* $v \otimes w \mapsto w \otimes v$[7] *We say that* $\mathcal{A}$ *is symmetric monoidal if* $\tau_{\mathbf{m},\mathbf{n}}\mathcal{A} = \tau_{\mathbf{m}\mathcal{A},\mathbf{n}\mathcal{A}}$.

Definition 2.4 (Monoidal Natural Transformations)**.** *Let* $\mathcal{A}, \mathcal{B} : \mathbf{2Cob} \to \mathbf{Vect}_{\Bbbk}$ *be symmetric monoidal functors. We say that a natural transformation* $u : \mathcal{A} \to \mathcal{B}$ *is monoidal if for every* $\mathbf{m}, \mathbf{n} \in \mathbf{2Cob}$ *we have* $u_{\mathbf{m}} \otimes u_{\mathbf{n}} = u_{\mathbf{m}\amalg\mathbf{n}}$ *and* $u_{\mathbf{0}} = \mathrm{id}_{\Bbbk}$.

We conclude having defined the following category (details are left to the reader).

Theorem 2.4. *The symmetric monoidal functors from* **2Cob** *to* $\mathbf{Vect}_{\Bbbk}$ *form a category (with objects such functors, arrows monoidal natural transformations, and composition defined as usual) which we denote* $\mathbf{Repr}_{\Bbbk}(\mathbf{2Cob}) \equiv \mathbf{SymMonCat}(\mathbf{2Cob}, \mathbf{Vect}_{\Bbbk})$.

3 The Atiyah-Segal Axioms

We now present the axiomatic formulation of TQFTs introduced by Atiyah in [2].

Definition 3.1 (Topological Quantum Field Theory)**.** *Let* M *be an oriented* n*-manifold, let* Σ *be a closed submanifold of* M *that is a connected component of* ∂M*. An* n*-dimensional TQFT is a rule* $\mathcal{A}$ *which to each* n*-1 dimensional* Σ *associates a vector space* $\Sigma\mathcal{A}$*, and to each oriented cobordism* $M : \Sigma_0 \to \Sigma_1$ *associates a linear map* $M\mathcal{A}$ *from* $M : \Sigma_0\mathcal{A}$ *to* $\Sigma_1\mathcal{A}$*, such that the following axioms hold.*

A1: Two equivalent cobordisms must have the same image, that is:

$$M \cong M' \implies M\ \mathcal{A} = M'\mathcal{A}$$

A2: The cylinder on Σ, must be sent to $\Sigma\mathcal{A}$.

[6]A simple example: T is diffeomorphic to a disjoint union of cylinders as a manifold, but not equivalent to the disjoint union of cylinders as a cobordism

[7]Recall the universal property of the tensor product: for vector spaces A and B, their tensor product A $\otimes$ B is a vector space with a bilinear map $\otimes : (a, b) \to a \otimes b$ from $A \times B$ to $A \otimes B$ such that for every bilinear map $f : A \times B \to C$ there is a unique linear map $h : A \otimes B \to C$ such that $f = h \circ \otimes$., forming a familiar commutative diagram.

A3: (Composition of Linear Maps). Given a decomposition $M = M'M''$,

$$M\mathcal{A} = (M'\mathcal{A}) \circ (M''\mathcal{A})$$

A4: (Multiplicative). Disjoint union goes to tensor product[8] : if $\Sigma = \Sigma' \coprod \Sigma''$ then $\Sigma\mathcal{A} = \Sigma = \Sigma'\mathcal{A} \otimes \Sigma''\mathcal{A}$. This holds for cobordisms. That is, if $M : \Sigma_0 \to \Sigma_1$ is the disjoint union of $M' : \Sigma_0' \to \Sigma_1'$ and $M'' : \Sigma_0' \to \Sigma_1''$ then $M\mathcal{A} = M'\mathcal{A} \otimes M''\mathcal{A}$.

A5: (Unital). The empty manifold $\Sigma = \emptyset$ must be sent to the ground field $\Bbbk$. It directly follows that the empty cobordism, which is the cylinder over $\Sigma = \emptyset$, is sent to $\mathrm{id}_{\Bbbk}$.

Axioms 1-3 establish the functoriality of $\mathcal{A} : \mathbf{nCob} \to \mathbf{Vect}_{\Bbbk}$, whereas axioms 4 and 5 say that $\mathcal{A}$ is a symmetric monoidal functor (so we may say $\mathbf{2TQFT}_{\Bbbk} := \mathbf{Repr}_{\Bbbk}(\mathbf{2Cob})$ is the category of 2TQFTs). An essential property of TQFTs is that they compute *topological invariants of manifolds*: if M is an n-manifold without a boundary, then we have a cobordism from the empty (n-1)-manifold to itself, so $\mathcal{A}$ associates to it a linear map $\Bbbk \to \Bbbk$, which is a constant—a topological invariant of the manifold.

4 Commutative Frobenius Algebras

We briefly review the theory of Frobenius algebras, which have been historically well-studied in representation and module theory [9], in working towards our ultimate goal of introducing the correspondence between TQFTs and commutative Frobenius Algebras[9].

Definition 4.1 ($\Bbbk$-algebra)**.** *A $\Bbbk$-algebra is a $\Bbbk$-vector space A along with two $\Bbbk$-linear maps called the multiplication and unit maps respectively:*

$$\mu : A \otimes A \to A \quad \eta : \Bbbk \to A$$

such the following three diagrams commute, defining id_a as the identity linear map and the diagonal maps without labels are scalar multiplication, which are canonical isomorphisms. The axioms (associativity and unity conditions) in terms of the elements of A follow: $(xy)z = x(yz)$, $1x = x = x1$.

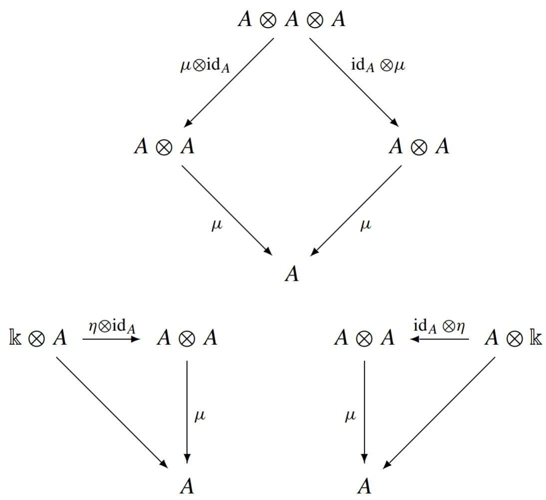
Definition 4.2 (Frobenius algebras)**.** *A Frobenius algebra is a finite-dimensional $\Bbbk$-algebra equipped with a linear functional* [10] *$\epsilon \to \Bbbk$ whose nullspace contains no nontrivial left ideals. The functional $\epsilon \in A^*$ is called the Frobenius form.*

In particular, each Frobenius form determines an associative non-degenerate pairing $\beta : A \otimes A \to \Bbbk$, which induces two $\Bbbk$-linear left and right isomorphisms between A and its dual space A^*, $\beta_L : A \tilde{\to} A^*$ and $\beta_R : A \tilde{\to} A^*$.

[8] See previous footnote.

[9] In our context, we note importantly that being a Frobenius Algebra pertains to having a Frobenius *form*, which is distinct from a property; this corresponding form is part of the Frobenius *structure* which is of interest to us.

[10] Recall the definition of linear functional, which is a linear map from a vector space A to the ground field $\Lambda : A \to \Bbbk$ and defines a hyperplane in A, $Null(\Lambda) := \{x \in A | x\Lambda = 0\}$.



These are the salient features of the Frobenius structure. We briefly remark by stating that having having no nontrivial left ideals in $Null(\epsilon)$ is equivalent to having no nontrivial principal left ideals in $Null(\epsilon)$, by taking a non-zero element and letting that element generate a principal ideal, so our condition can be written as $(Ay)\epsilon = 0 \implies y = 0$. We say that a Frobenius algebra (A, ϵ) is commutative if the twist map $\sigma : A \otimes A \to A \otimes A$ is such that $\sigma\mu = \mu$.

Theorem 4.1. *A Frobenius algebra homomorphism $\phi : (A, \epsilon) \to (A', \epsilon)$ between two Frobenius algebras is a homomorphism which is at the same time a coalgebra homomorphism (see Appendix) preserving the Frobenius form, $\epsilon = \phi\epsilon'$. Therefore, the category of Frobenius algebras $\mathbf{FA}_{\Bbbk}$ includes the Frobenius algebras over $\Bbbk$ as objects and Frobenius algebra homomorphisms as morphisms. $\mathbf{cFA}_{\Bbbk}$ is the subcategory of commutative Frobenius algebras.*

Frobenius algebras appear in many disciplines because they essentially define a representation of a general topological structure whose axioms can be given as graphs, for instance, or in our case, topological surfaces.

5 2d TQFTs and Commutative Frobenius Algebras

We are now ready to sketch a proof of the following result.

Theorem 5.1. *For any ground field $\Bbbk$, there is an equivalence*

$$\mathbf{2TQFT}_{\Bbbk} \simeq \mathbf{cFA}_{\Bbbk},$$

by sending each TQFT to its valuation on the circle. Here $\mathbf{cFA}_{\Bbbk}$ is the category of commutative Frobenius $\Bbbk$-algebras.

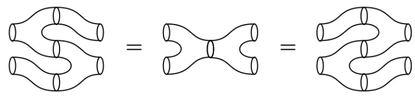
Proof. By $\mathbf{C} \simeq \mathbf{D}$ we mean that there exist $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{E}$, and natural isomorphisms $\alpha : F \circ G \to \mathrm{id}_{\mathbf{C}}$, $\beta : \mathrm{id}_{\mathbf{D}} \to G \circ F$. Here we show that there is a map on objects $\mathbf{2TQFT}_{\Bbbk} \to \mathbf{cFA}_{\Bbbk}$, while sketching the induced map on arrows. Ultimately we are forced to leave the details to the reader, and to [5] pp.171. Recall $\mathbf{2TQFT}_{\Bbbk} = \mathbf{Repr}_{\Bbbk}(\mathbf{2Cob}) = \mathbf{SymMonCat}(\mathbf{2Cob}, \mathbf{Vect}_{\Bbbk})$. Since a symmetric monoidal functor on **2Cob** is defined by its valuation on the skeleton and the generators of the source category, we may construct the above correspondence as follows. Recall that the skeleton of **2Cob** was given by $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \cdots\}$ where $\mathbf{n}$ is a disjoint union of n copies of the circle $\mathbf{1}$. Now let $\mathcal{A} \in \mathbf{2TQFT}_{\Bbbk}$ and let $A := \mathbf{1}\mathcal{A}$. Since $\mathcal{A}$ is symmetric monoidal, it follows that $\mathbf{n}\mathcal{A} = A^n := A \otimes A \otimes \cdots \otimes A$. It is also forced that the cylinder on $\mathbf{1}$ is mapped to $\mathrm{id}_A : A \to A$ and that the twist generator of **2Cob** is mapped to the twist $\sigma : A^2 \to A^2$ given by swapping vectors. Now let us denote the images of the other generators of **2Cob** as follows (using the labeling in 2.2): $\eta : \Bbbk \to A$ is the image of $(\mathbf{0}, \mathbf{1})$, $\mu : A^2 \to A$ is the image of $(\mathbf{2}, \mathbf{1})$, $\epsilon : A \to \Bbbk$ is the image of $(\mathbf{1}, \mathbf{0})$, and $\delta : A \to A^2$ is the image of $(\mathbf{1}, \mathbf{2})$. This defines $\mathcal{A}$ entirely, as noted initially. So a 2TQFT is prescribed by a vector space A, along with linear maps between tensor powers of A that satisfy analogues the relations listed in A. But these are exactly the data (regarding μ as multiplication, δ as comultiplication, η as the unit and ϵ as the counit) defining a commutative Frobenius algebra as noted in A.3! It follows that A is a commutative Frobenius algebra. The map taking commutative Frobenius algebras to 2TQFTs are defined in a similar manner, and its construction is left to the reader.

If $u : \mathcal{A} \to \mathcal{B}$ is a monoidal natural transformation between 2TQFTs, we find that u is completely specified by the linear map $\mathbf{1}\mathcal{A} \to \mathbf{1}\mathcal{B}$. The naturality of u will then force the map $\mathbf{1}\mathcal{A} \to \mathbf{1}\mathcal{B}$ to be a Frobenius algebra homomorphism. A similar argument applies for taking Frobenius homomorphisms to 2TQFTs. □

A Relations for 2Cob and Frobenius Algebras

Theorem A.1. *The following relations hold on* **2Cob***.*

(a) The Unit and Co-unit Relations



(b) The Frobenius Relation

(c) The Commutativity Relation

Figure A.1: A Few Relations on **2Cob**. Illustrations taken from [5]

Proof. Note that in each relation, the cobordisms involved have the same topology type – i.e. the same Euler characteristic. Recall that $\chi = 2 - 2g - (m + n)$ where g is the genus of the surface ($g = 0$ here), m is the number of out-boundaries, and n the number of in-boundaries. Therefore, by the classification of connected oriented manifolds (with boundary), we see that the above relations hold. □

We also present the following theorems without proof. See [5] pp.28 for details.

Theorem A.2. *Every Frobenius algebra* (A, ϵ) *is also a co-algebra. I.e. to each Frobenius algebra* (A, ϵ)*, we can associate a unique coassociative comultiplication* $\delta : A \to A \otimes A$ *(with counit* $\epsilon : A \to \Bbbk$*) satisfying the Frobenius relation (here whenever we say co-, we just mean the dual notion).*

Theorem A.3. *Let* $A \in \mathbf{Vect}_{\Bbbk}$ *equipped with a multiplication* $\mu : A \otimes A \to A$*, unit* $\eta : \Bbbk \to A$*, comultiplication* $\delta : A \to A \otimes A$*, and counit* $\epsilon : A \to \Bbbk$*, satisfy the commutativity (i.e. that the canonical twist map* $\sigma : A \otimes A \to A \otimes A$ *is such that* $\tau\mu = \mu$*) and Frobenius relation (represented by (b) in A.1). Then* A *equipped with* ϵ *is a commutative Frobenius algebra.*

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